$$\begin{tikzcd}
\to H_{i+1}(M, M\setminus\partial U) \arrow[r] & H_i(U \amalg M\setminus\bar{U}, M\setminus\bar{U}) \arrow[r] & H_i(M, M\setminus\bar{U}) \arrow[r] & {} \\
& H_i(U) \arrow[r] \arrow[u, "\cong\ \text{excision}"] & H_i(M) \arrow[u] & \\
& {[\varphi]} \arrow[r, mapsto] & {[\varphi]} &
\end{tikzcd}$$

($H_{i+1}(M, M\setminus\partial U)$: 0 for $i \geq n$; $H_i(M, M\setminus\bar{U})$: 0 for $i > n$)

$\Rightarrow H_i(U) = 0 \quad \forall i > n.$

It remains to deal with the case $i = n$.

Let $x_0 \in M\setminus\bar{U}$ (recall that M is non-cpt. and thus $\emptyset \neq M\setminus\bar{U}$)

$res_{x_0}[\varphi] = 0$ in $H_n(M, M\setminus x_0)$ since $[\varphi]$ can be regarded as a class in $H_n(M\setminus x_0)$

$\Rightarrow$ the section $M \ni x \longmapsto res_x[\varphi]$ vanishes by connectedness of M.

$\Rightarrow$ the img of $[\varphi]$ in $H_n(M, M\setminus\bar{U}) \cong \Gamma_{\bar{U}}\left(\begin{smallmatrix} M_R \\ \downarrow \\ M\end{smallmatrix}\right)$ vanishes

Looking at the diagram above we thus conclude that $[\varphi] = 0$ in $H_n(U)$.

Torsion freeness follows as before from the UCT. □

Cohomology with compact support & Duality for noncpt mfs.

28.05.201[illegible]

Thm. (Poincaré duality) M n-mf. conn. cpt. R-orientable.

Then $[M]\cap - : H^\ell(M; R) \xrightarrow{\cong} H_{n-\ell}(M; R)$.

For the pf. we need a bit more than just cptness: we need a thy of chlg w/ cpt support.

Let M be an n-mf., $K \subseteq M$ cpt subset. Then we have

$$H_n(M, M\setminus K; R) \xrightarrow{\cong} \Gamma_K(M_R), \quad \alpha \longmapsto (x \mapsto res_x(\alpha))_{x\in M}$$

where $\Gamma_K(M_R) = \{$c. sections of $M_R \xrightarrow{p_R} M$ defined on $K\}$,

$$M_R = \coprod_{x\in M} H_n(M, M\setminus x; R).$$

i.e. given a c. section $K \xrightarrow{s} M_R$ $\exists!\, \alpha \in H_n(M, M\setminus K; R)$ s.t. $\forall x\in K$: $res_x(\alpha) = s(x)$.

$$H^\ell(M, M\setminus K; R) \overset{def}{=} H^\ell(C^*(M, M\setminus K; R))$$

$$\varphi \in C^\ell(M, M\setminus K; R) \overset{def}{\Longleftrightarrow} \underbrace{\varphi : C_\ell(M; \mathbb{Z}) \longrightarrow R \quad \& \quad \varphi|_{C_\ell(M\setminus K; \mathbb{Z})} = 0}_{\text{i.e. "functions supported in } K\text{"}}$$

Ex. $M = \mathbb{R}^n$, $K = x_0$. Consider the collection of functions $\varphi : C_\ell(\mathbb{R}^n; \mathbb{Z}) \longrightarrow \mathbb{Z}$ for which $\varphi(\sigma) = 0$ unless $\operatorname{Im}\sigma \subseteq \{x_0\}$ (i.e. unless $\sigma \equiv \text{const}_{x_0}$)

$\varphi \in C^0(\mathbb{R}^n; \mathbb{Z}) = \operatorname{Hom}(C_0(\mathbb{R}^n; \mathbb{Z}), \mathbb{Z})$. Let $\sigma_x : \Delta^0 \longrightarrow \mathbb{R}^n$, $* \longmapsto x$, $\forall x \in \mathbb{R}^n$

$\varphi(\sigma_{x_0}) = 1$ and $\varphi(\sigma_x) = 0$ $\forall x \neq x_0$.

$\delta\varphi \in C^1(\mathbb{R}^n; \mathbb{Z})$, $\sigma \in C_1(\mathbb{R}^n; \mathbb{Z})$: $(\delta\varphi)(\sigma) = \varphi(\sigma(0,1)) - \varphi(\sigma(1,0))$

Take σ_{xx_0} starting at x_0 and ending at $x \neq x_0$

$(\delta\varphi)(\sigma_{xx_0}) = 1$, $\operatorname{Im}\sigma_{xx_0} \nsubseteq K = \{x_0\}$

This example shows that the above statement that $C^\ell(M, M\setminus K)$ is the set of functions supported in K is not quite correct; the actual set of those functions would not be a subcomplex.

Ex. $H_n(\mathbb{R}^n; \mathbb{Z}) = 0$ $\forall n \geq 1$ Poincaré duality fails:

$$0 = H^n(\mathbb{R}^n; \mathbb{Z}) \underset{\ncong}{\xrightarrow{\;0\;}} H_{n-n}(\mathbb{R}^n; \mathbb{Z}) = H_0(\mathbb{R}^n; \mathbb{Z}) = \mathbb{Z}$$

Def. Let X be a space. A compactly supported cochain of dimension ℓ is a $\varphi \in C^\ell(X; R)$ s.t. $\exists K \subseteq X$ cpt. s.t. $\varphi|_{C_\ell(X\setminus K; \mathbb{Z})} = 0$, i.e. we have $\varphi \in C^\ell(X, X\setminus K; R)$. The collection of such cochains is $C^\ell_c(X; R)$.

Note. Prof. Putchkoria does not include T2 in the def. of cptness.

$C^*_c(X; R) \subseteq C^*(X; R)$ is a sub-cochain complex.

$\varphi|_{C_\ell(X\setminus K; \mathbb{Z})} = 0 \implies \delta\varphi|_{C_\ell(X\setminus K; \mathbb{Z})} = 0$, i.e. $C^*(X, X\setminus K; R)$ is a cochain cx.

$$\left. C^*_c(X; R) = \bigcup_{\substack{K \subseteq X \\ \text{cpt.}}} \underbrace{C^*(X, X\setminus K; R)}_{\subseteq C^*(X; R)} \right\} \subseteq C^*(X; R)$$

Def. The compactly supported cohomology of X is the cohomology of $C^*_c(X;R)$: $\qquad H^\ell_c(X;R) := H^\ell(C^*_c(X;R))$.

How may we compute $H^\ell_c(X;R)$ in terms of $H^*(X, X\setminus K; R)$?

First we do some algebra: colimits.

Def. Let J be a poset. We call J directed if

$$\forall \alpha,\beta \in J \quad \exists \gamma \in J: \quad \alpha \le \gamma, \ \beta \le \gamma.$$

Def. An J-indexed direct system of ab. groups is a collection

$$\{G_\alpha, f_{\alpha\beta} \mid \alpha \in J, \ \beta \geq \alpha\} \qquad \text{s.t.}$$

- $\forall \alpha \in J$ we have an ab. gp. G_α
- $\forall \alpha,\beta \in J$, $\alpha \le \beta$ we have a homomorphism $G_\alpha \xrightarrow{f_{\alpha\beta}} G_\beta$

subject to the conditions

1) $f_{\alpha,\alpha} = id_{G_\alpha}$
2) $\alpha \le \beta \le \gamma \Rightarrow f_{\alpha\gamma} = f_{\beta\gamma} \circ f_{\alpha\beta}$, i.e.

$$\begin{tikzcd} G_\alpha \arrow[rr, "f_{\alpha\gamma}"] \arrow[dr, "f_{\alpha\beta}"'] & \circlearrowleft & G_\gamma \\ & G_\beta \arrow[ur, "f_{\beta\gamma}"'] & \end{tikzcd}$$

Note. Any poset can be considered as a category. objects are the elts of the poset, $\mathrm{Mor}(\alpha,\beta) = \begin{cases} \emptyset & \alpha \nleq \beta \\ * & \alpha \le \beta \end{cases}$

Then we may consider J as a category and a direct system is just a functor $J \longrightarrow Ab$.

Rmk. An $J = (\mathbb{N}, \le)$-directed system on $(\mathbb{N}, \le)$ is a sequence

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \longrightarrow \dots$$

with $f_{nm} = f_{m-1} \circ \dots \circ f_n : A_n \xrightarrow{f_n} \dots \xrightarrow{f_{m-1}} A_m \quad \forall n \le m$.

Ex. $\mathbb{Z}/p \xrightarrow{\cdot p} \mathbb{Z}/p^2 \xrightarrow{\cdot p} \mathbb{Z}/p^3 \longrightarrow \dots$

Def. Given a directed poset J and direct system $\{G_\alpha, f_{\alpha\beta}\}$ define

$$\operatorname*{colim}_{\alpha \in J} G_\alpha := \bigoplus_{\alpha \in J} G_\alpha \Big/ (x - f_{\alpha\beta}(x)) \ \forall x \in G_\alpha, \ \alpha \le \beta$$

Note that as usual, we abuse the notation $x \in G_{\alpha_0} \hookrightarrow \bigoplus_{\alpha \in J} G_\alpha$.

$$\begin{tikzcd} G_\alpha \arrow[r, hook] \arrow[dr, "\iota_\alpha"'] & \bigoplus_{\alpha\in J} G_\alpha \arrow[d] \\ & \operatorname{colim}_{\alpha\in J} G_\alpha \end{tikzcd}$$

$\iota_\alpha(x) = [x] = [(x_\beta)]$ where $x_\beta = \begin{cases} 0 & \beta \neq \alpha \\ x & \beta = \alpha \end{cases}$

Properties.

(PO)

$$\begin{tikzcd} \overset{x}{G_{\alpha_0}} \arrow[rr, "\iota_{\alpha_0}"'] \arrow[d, "f_{\alpha_0\alpha_1}"] \arrow[drr, phantom, "\circlearrowright"] & & \operatorname{colim}_{\alpha\in J} G_\alpha \\ \underset{f_{\alpha_0\alpha_1}(x)}{G_{\alpha_1}} \arrow[urr, "\iota_{\alpha_1}"'] & & \end{tikzcd}$$

$[x] = [f_{\alpha_0\alpha_1}(x)]$ since we quotient out by their difference.

Universal property: given any ab gp C with maps $G_\alpha \xrightarrow{\varphi_\alpha} C$ $\forall \alpha \in J$

s.t. $\forall \alpha \le \beta$:

$$\begin{tikzcd} G_\alpha \arrow[r, "\varphi_\alpha"] \arrow[d, "f_{\alpha\beta}"'] \arrow[dr, phantom, "\circlearrowright"] & C \\ G_\beta \arrow[ur, "\varphi_\beta"'] & \end{tikzcd}$$

then $\exists!\ \operatorname{colim}_{\alpha\in J} G_\alpha \xrightarrow{\varphi} C$ s.t. $\varphi \circ \iota_\alpha = \varphi_\alpha$, $\forall \alpha \in J$.

(This follows from the univ. prop. of $\oplus$.)

Alternative description of $\operatorname{colim}_{\alpha\in J} G_\alpha$: (here we really use that J is directed)

$$\operatorname{colim}_{\alpha\in J} G_\alpha \cong \coprod_{\alpha \in J} G_\alpha \Big/ \sim$$

where for $a \in G_\alpha$, $b \in G_\beta$ we have

$$a \sim b \iff \exists \gamma \in J: \alpha \le \gamma,\ \beta \le \gamma,\ f_{\alpha\gamma}(a) = f_{\beta\gamma}(b).$$

Group structure: $a \in G_\alpha$, $b \in G_\beta$: $[a] + [b] = [f_{\alpha\gamma}(a) + f_{\beta\gamma}(b)]$ for some $\alpha \le \gamma$, $\beta \le \gamma$.

Cor. $x \in G_\alpha$. Then $0 = [x] \in \operatorname{colim}_{\alpha\in J} G_\alpha$ iff $\exists \gamma \ge \alpha$: $f_{\alpha\gamma}(x) = 0 \in G_\gamma$.

$$A_0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \dots \longrightarrow \operatorname{colim}_{n\in\mathbb{N}} A_n$$

Ex. $\operatorname{colim}_{n\in\mathbb{N}} \mathbb{Z}/p^n = \mathbb{Z}/p^\infty = \mathbb{Z}[\tfrac{1}{p}]/\mathbb{Z} \cong \mathbb{Q}/\mathbb{Z}_p \cong \mathbb{Q}_p/\mathbb{Z}_p$ Prüfer group

We generalise this from ab. gps to chain complexes.

Def. A direct system (J-indexed) of cochain complexes is the following data:

- $\forall \alpha \in J$: C_α cochain complex
- $\forall \alpha \le \beta$: $f_{\alpha\beta}: C_\alpha \longrightarrow C_\beta$

such that $f_{\alpha\alpha} = id_{C_\alpha}$, $f_{\alpha\gamma} = f_{\beta\gamma} \circ f_{\alpha\beta}$.

Now $\operatorname{colim}_{\alpha\in J} C_\alpha$ as a cochain complex is defined as:

$$\left(\operatorname{colim}_{\alpha\in J} C_\alpha\right)^n = \operatorname{colim}_{\alpha\in J} C_\alpha^n \qquad n\in\mathbb{Z}$$

$$\begin{tikzcd} C_\alpha^n \arrow[r, "d_\alpha^n"] \arrow[d, "f_{\alpha\beta}"'] \arrow[dr, phantom, "\circlearrowright"] & C_\alpha^{n+1} \arrow[r, "\iota_\alpha^{n+1}"] \arrow[d, "f_{\alpha\beta}^{n+1}"] & \operatorname{colim}_{\alpha\in J} C_\alpha^{n+1} \\ C_\beta^n \arrow[r, "d_\beta^n"] & C_\beta^{n+1} \arrow[ur, "\iota_\beta^{n+1}"'] & \end{tikzcd}$$

Differential: $\exists!\, d: \operatorname{colim}_{\alpha\in J} C_\alpha^n \xrightarrow{d} \operatorname{colim}_{\alpha\in J} C_\alpha^{n+1}$, $\quad d\circ\iota_\alpha^n = \iota_\alpha^{n+1}\circ d$, $\quad d^2=0.$

$$[c] \longmapsto [d(c)]$$

Back to cpt supp chg.

Given a space X, the set of all compact subsets of X is a directed poset : $\forall K, K'$: $K\subseteq K\cup K'\supseteq K'$

For $K\subseteq L$ compacts subsets of X: $X\setminus L\subseteq X\setminus K$

$$C^*(X, X\setminus K; R) \xrightarrow[\substack{\iota^*_{KL}=\iota^* \\ (\text{a.o.n.})}]{} C^*(X, X\setminus L; R)$$

Prop. (Exercise) $\left(C^*(X, X\setminus K; R), \iota^*_{KL}\right)$ is a directed system of cochain cxs.

Prop. (Exc.) 1) $\operatorname{colim}_K C^*(X, X\setminus K; R) \cong C_c^*(X;R)$

2) Given a directed system C_α of cochain complexes,

$$H^\ell\left(\operatorname{colim}_{\alpha\in J} C_\alpha\right) \xleftarrow{\cong} \operatorname{colim}_{\alpha\in J} H^\ell(C_\alpha).$$

Cor. The canonical map

$$\operatorname{colim}_K H^\ell(X, X\setminus K; R) \xrightarrow{\cong} H_c^\ell(X;R) \quad \text{is an isomorphism.}$$

Ex. Compute $H_c^\ell(\mathbb{R}^n, G) \cong \operatorname{colim}_K H^\ell(X, X\setminus K, G)$

Prop. (Excercise) $\mathcal{F}\subseteq J$ is cofinal if $\forall \alpha\in J\ \exists \gamma\in\mathcal{F}: \alpha\le\gamma$.

$G_\gamma \xrightarrow{\iota_\gamma} \operatorname{colim}_{\alpha\in J} G_\alpha$ induce by the univ prop a unique

$$\operatorname{colim}_{\gamma\in\mathcal{F}} G_\alpha \longrightarrow \operatorname{colim}_{\alpha\in J} G_\alpha$$

Now in our example, let J be the poset of balls of radius n.

This is cofinal in all compacts.

$$\Rightarrow \quad H^l_c(\mathbb{R}^n, G) \cong \operatorname*{colim}_{m\in\mathbb{N}} H^l(\mathbb{R}^n, \mathbb{R}^n \setminus B(m), G)$$

where $B(m)$ is the ball of radius m.

$$H^l(\mathbb{R}^n, \mathbb{R}^n \setminus B(m), G) \cong \begin{cases} 0 & l \neq n \\ G & l = n \end{cases}$$

$$H^l_c(\mathbb{R}^n, G) = 0 \quad \text{unless } n = l$$

$$\begin{tikzcd} H^l_c(\mathbb{R}^n, \mathbb{R}^n \setminus B(m); G) \arrow[r] \arrow[d, phantom, "\wr\wr"] & H^n_c(\mathbb{R}^n, \mathbb{R}^n \setminus B(m); G) \arrow[d, phantom, "\wr\wr"] \\ G \arrow[r, "id"] & G \end{tikzcd}$$

$$\operatorname*{colim}_{m\in\mathbb{N}} H^n_c(\mathbb{R}^n, \mathbb{R}^n \setminus B(m); G) = \operatorname*{colim}_{m\in\mathbb{N}} G \cong G \quad \Rightarrow \quad H^n_c(\mathbb{R}^n; G) \cong G$$

Prop. (Exercise) If a directed poset J has a largest element $\gamma \in J$ (i.e. $\forall \alpha \in J: \alpha \leq \gamma$) then $\operatorname*{colim}_{\alpha\in J} G_\alpha \cong G_\gamma$.

$$H^*_c(X; R) = \operatorname*{colim}_{K} H^*(X, X\setminus K; R) \underset{\text{for } X \text{ cpt.}}{\cong} H^*(X, X\setminus X; R) = H^*(X; R)$$

So compactly supported chg. for cpts is just the usual chg.

30.05.2018

Let M be an R-orientable n-manifold. Then $\forall K \subset M$ cpt. subset we have

$$\cap: H_n(M, M\setminus K; R) \otimes_R H^l(M, M\setminus K; R) \longrightarrow H_{n-l}(M; R)$$

Note that $\{\mu_x \in H_n(M, M\setminus x; R) \mid x \in M\}$ is an orientation of M (def.) and by a previous Lemma $\exists!\, \mu_K \in H_n(M, M\setminus K; R)$ s.t. $res_x(\mu_K) = \mu_x \ \forall x \in K$.

If $K \subseteq L \subseteq M$ are cpt subsets of M then $K \xhookrightarrow{\imath} L$ induces

$$H_n(M, M\setminus L; R) \xrightarrow{\imath_*} H_n(M, M\setminus K; R).$$

The uniqueness in the last statement implies $\imath_*(\mu_L) = \mu_K$ since they restrict to the same thing.

"I don't want to mix up some formulas with actual mathematics."

Recall: $f_*(\alpha)\cap\varphi = f_*(\alpha\cap f^*(\varphi))$

<u>Claim.</u>
$$\begin{tikzcd} H^\ell(M, M\setminus K; R) \arrow[rr, "\mu_K\cap -"] \arrow[d, "i^*"'] \arrow[drr, phantom, "\circlearrowleft"] & & H_{n-\ell}(M;R) \\ H^\ell(M, M\setminus L; R) \arrow[urr, "\mu_L\cap -"'] & & \end{tikzcd}$$

Using the formula above:

$$\underbrace{i_*(\mu_L)}_{\mu_K}\cap[\varphi] = id_*(\mu_L\cap i^*[\varphi]) = \mu_L\cap i^*[\varphi] \quad \Rightarrow \text{Claim.}$$

Thus we have a map D_M by the univ. prop:

$$\begin{tikzcd} \underbrace{\operatorname{colim}_{K\subseteq M} H^\ell(M, M\setminus K; R)}_{H^\ell_c(M;R)} \arrow[rr, "D_M"] & & H_{n-\ell}(M;R) \\ H^\ell(M, M\setminus K; R) \arrow[u, "i_K"] \arrow[urr, "\mu_K\cap -"'] & & \end{tikzcd}$$

<u>Thm.</u> (Poincaré duality for not necessarily cpt manifolds)

For R-orientable n-manifolds M D_M is an isomorphism.

This clearly generalises the previous version of P. duality.

<u>Prop.</u> Let M be an R-orientable n-dim mf. and suppose $M = U\cup V$ where U, V are open. Then there is a commutative diagram:

$$\begin{tikzcd} \cdots \arrow[r] & H^\ell_c(U\cap V) \arrow[r, "(i_{1*}, i_{2*})"] \arrow[d, "D_{U\cap V}"] & H^\ell_c(U)\oplus H^\ell_c(V) \arrow[r, "(+i_{U*}, -i_{V*})"] \arrow[d, "D_U\oplus D_V"] & H^\ell_c(U\cup V) \arrow[r, "(-1)^{\ell+1}\delta", "(M)"'] \arrow[d, "D_{U\cup V}"] & H^{\ell+1}_c(U\cap V) \arrow[r] \arrow[d, "D_{U\cap V}"] & \cdots \\ \cdots \arrow[r] & H_{n-\ell}(U\cap V) \arrow[r, "(+-)"] & H_{n-\ell}(U)\oplus H_{n-\ell}(V) \arrow[r, "(++)"] & H_{n-\ell}(U\cup V) \arrow[r, "\partial"] & H_{n-\ell-1}(U\cap V) \arrow[r] & \cdots \end{tikzcd}$$

where both rows are exact, and the bottom row is the MV sequence in homology.

$i_1: U\cap V \hookrightarrow U$, $i_2: U\cap V\hookrightarrow V$,
$i_U: U\hookrightarrow U\cup V$, $i_V: V\hookrightarrow U\cup V$.

Pf: For a space X and $U \xhookrightarrow{i} X$ open we have

$$\begin{tikzcd} H^*_c(U) \arrow[r, "i_*"] \arrow[d, phantom, "\cong" rotate=90] & H^*_c(X) \arrow[d, phantom, "\cong" rotate=90] \\ \operatorname{colim}_{K\subseteq U} H^*(U, U\setminus K) & \operatorname{colim}_{X\subseteq U} H^*(X, X\setminus K) \end{tikzcd}$$

Excision $\Rightarrow$ $H^*(X, X\setminus K) \xrightarrow{\cong} H^*(U, U\setminus K)$ if $K \subseteq U$.

$$\Rightarrow H^*_c(U) \cong \operatorname{colim}_{K\subseteq U} H^*(X, X\setminus K) \xrightarrow{i_*} \operatorname{colim}_{K\subseteq X} H^*(X, X\setminus K)$$

Moreover if $U \xhookrightarrow{i} V \xhookrightarrow{j} X$ then $(ji)_* = j_* \circ i_*$, i.e. we have functoriality.

Let $K\subseteq U$, $L \subseteq V$ be cpt. $\Rightarrow$ $K\cup L \subseteq U\cup V = M$, $K\cap L \subseteq U\cap V$.

$$H^l_c(U) = \operatorname{colim}_{K\subseteq U} H^l(U, U\setminus K) \cong \operatorname{colim}_{K\subseteq U} H^l(M, M\setminus K)$$

$$H^l_c(V) = \operatorname{colim}_{L\subseteq V} H^l(V, V\setminus L) \cong \operatorname{colim}_{L\subseteq V} H^l(M, M\setminus L)$$

$M\setminus(K\cup L) \subseteq M\setminus K$ and $M\setminus(K\cup L)\subseteq M\setminus L$. We get induced maps

$$\begin{tikzcd} H^l(M, M\setminus K) \arrow[r, "i_K"] & H^l(M, M\setminus(K\cup L)) \arrow[d, equal] \\ H^l(M, M\setminus L) \arrow[r, "i_L"] & H^l(M, M\setminus(K\cup L)) \end{tikzcd}$$

$$H^l(M, M\setminus K) \oplus H^l(M, M\setminus L) \xrightarrow{(i_K, -i_L)} H^l(M, M\setminus(K\cup L))$$

Pass to colimits along K and L:

$$\begin{tikzcd} H^l_c(U)\oplus H^l_c(V) = \operatorname{colim}_{K\subseteq U} H^l(M, M\setminus K) \oplus \operatorname{colim}_{L\subseteq V} H^l(M, M\setminus L) \arrow[d] \arrow[dr] & \\ H^l_c(M) \cong \operatorname{colim}_{K\subseteq U, L\subseteq V} H^l(M, M\setminus(K\cup L)) & \end{tikzcd}$$

using cofinality.

$$H^l(M, M\setminus(K\cap L)) \longrightarrow H^l(M, M\setminus K)\oplus H^l(M, M\setminus L)$$

Pass to colim along K, L, colim commutes with $\oplus$

$$\begin{tikzcd} \operatorname{colim}_{K\subseteq U, L\subseteq V} H^l(M, M\setminus(K\cap L)) \arrow[r] \arrow[d, equal] & \operatorname{colim}_{K\subseteq U} H^l(M, M\setminus K) \oplus \operatorname{colim}_{L\subseteq V} H^l(M, M\setminus L) \arrow[d, equal] \\ H^l_c(U\cap V) \arrow[r] & H^l_c(U)\oplus H^l_c(V) \end{tikzcd}$$

Now we see that the maps involved in the Prop are colimits of certain maps for K, L. Thus we may deal with these maps and then pass to colim.

(directed colim is an exact functor)

$C \subset M$ cpt., $C \subseteq U$ open, $\mu_C \in H_n(M, M \setminus C)$

$$\underbrace{H_n(M, M\setminus C) \cong H_n(U, U\setminus C)}_{\text{from now on we identify these two groups}} \ni \mu_C \text{ is a singular chain in } U$$

Then it suffices to show that the following commutes:

$$\begin{array}{ccccccc}
\cdots \to H^\ell(M, M\setminus(K\cap L)) & \xrightarrow{(++)} & H^\ell(M,M\setminus K)\oplus H^\ell(M,M\setminus L) & \xrightarrow{(+-)} & H^\ell(M, M\setminus(K\cup L)) & \xrightarrow{(-1)^{\ell+1}\delta} & H^{\ell+1}(M, M\setminus(K\cap L)) \to \cdots \\
\downarrow \cong \text{exc.} & & \cong\downarrow \text{exc}\oplus(-\text{exc}) & & & (\#) & \cong \downarrow \text{exc.} \\
H^\ell(U\cap V, (U\cap V)\setminus(K\cap L)) & & H^\ell(U, U\setminus K)\oplus H^\ell(V, V\setminus L) & & \downarrow \mu_{K\cup L}\cap - & & H^{\ell+1}(U\cap V, (U\cap V)\setminus(K\cap L)) \\
\downarrow \mu_{K\cap L}\cap - & & \downarrow (\mu_K\cap -)\oplus(\mu_L\cap -) & & & & \downarrow \mu_{K\cap L}\cap - \\
\cdots \to H_{n-\ell}(U\cap V) & \xrightarrow{(+-)} & H_{n-\ell}(U)\oplus H_{n-\ell}(V) & \xrightarrow{(++)} & H_{n-\ell}(U\cup V) & \xrightarrow{\partial} & H_{n-\ell-1}(U\cap V) \to \cdots
\end{array}$$

The top row is the relative MV sequence for $(M, A = M\setminus K, B := M\setminus L)$.

The first two squares are easily seen to commute: this can be checked on the cocycle level. The painful part, as always, is the one involving the boundary homomorphisms. (#)

$[\varphi] \in H^\ell(M, M\setminus(K\cap L))$

$\mu_{K\cap L} \cap \text{exc} \circ \delta[\varphi] = ?$ this is the hard part

$$0 \to C^*(M, A+B) \to C^*(M,A)\oplus C^*(M,B) \to C^*(M, A\cap B) \to 0 \quad \text{sh. ex. sq}$$

$\psi \mapsto (\psi, \psi)$; φ_A, φ_B; φ

Recall: ψ if $\psi|_{C_*(A)} = 0 = \psi|_{C_*(B)}$ (but $\psi|_{C_*(A\cup B)}$ need not be 0)

$$H^\ell(C^*(M, A+B)) \underset{\text{exc}}{\cong} H^\ell(C^*(M, A\cup B)) = H^\ell(M, A\cup B)$$

For $\varphi \in C^*(M, A\cap B)$ find $\varphi_A \in C^*(M,A)$, $\varphi_B \in C^*(M,B)$ s.t. $\varphi = \varphi_A - \varphi_B$.

$(d\varphi_A, d\varphi_B) \in C^*(M,A)\oplus C^*(M,B)$.

$d\varphi_A - d\varphi_B = d\varphi = 0$ as φ is a cocycle $\Rightarrow d\varphi_A = d\varphi_B$

$\Rightarrow d\varphi_A$ gets sent to $(d\varphi_A, d\varphi_A) = (d\varphi_A, d\varphi_B) \Rightarrow [d\varphi_A] = \delta[\varphi]$.

$\Rightarrow \delta[\varphi] = [d\varphi_A + d\psi]$ in $H^*(M, A\cup B)$ where ψ is a cocycle with

$\psi|_{C_*(M,A)} = 0$, $\psi|_{C_*(M,B)} = 0$.

04.06.2018

There is a canonical morphism $H^*(M, A\cup B) \xrightarrow{\cong} H^*(M, A+B)$.

A priori $[d\varphi_A]$ sits in $H^*(M, A+B)$, this is why we need this error term $d\psi$.

$M = (U\setminus L) \cup (U\cap V) \cup (V\setminus K)$

By using the lemma about small simplices (Top I): there is a representative $\mu_{K\cup L} \in H_n(M, M\setminus(K\cup L))$ of the form

$$\alpha_{U\setminus L} + \alpha_{U\cap V} + \alpha_{V\setminus K}$$

where $\alpha_{U\setminus L}$, $\alpha_{U\cap V}$ and $\alpha_{V\setminus K}$ are singular chains in $U\setminus L$, $U\cap V$, $V\setminus K$, respectively.

Recall:

$$\begin{array}{ccl} H_n(M, M\setminus(K\cup L)) & \longrightarrow & H_n(M, M\setminus K) \\ \mu_{K\cup L} & & \mu_K \\ & \searrow & H_n(M, M\setminus L) \\ & & \mu_L \\ & \searrow & H_n(M, M\setminus(K\cap L)) \\ & & \mu_{K\cap L} \end{array}$$

i.e. μ_K, μ_L and $\mu_{K\cap L}$ can be rediscovered by restricting $\mu_{K\cup L}$.

$\alpha_{U\cap V}$ is a rep. for $\mu_{K\cap L}$: $\alpha_{U\setminus L}$ and $\alpha_{V\setminus K}$ are already zero in $C_n(M, M\setminus(K\cap L))$

Similarly: $\alpha_{U\setminus L} + \alpha_{U\cap V}$ is a rep. for μ_K,

$\alpha_{U\cap V} + \alpha_{V\setminus K}$ is a rep. for μ_L.

(Note that $[\alpha_{U\cap V}] \in H_n(U\cap V, U\cap V\setminus(K\cap L)) \xrightarrow{\cong} H_n(M, M\setminus(K\cap L))$.)

$\Rightarrow \mu_{K\cap L} \cap \operatorname{exc}\circ\delta[\varphi]$ is represented by $\alpha_{U\cap V} \cap (d\varphi_A + d\psi)$.

$\alpha_{U\cap V} \cap (d\varphi_A + d\psi)$ is a cycle in $U\cap V$.

We claim that $\alpha_{U\cap V} + d\psi$ is a boundary (and hence we can get rid of it)

$$d(\alpha_{U\cap V} \cap \psi) = (-1)^{\deg\psi}\Big(\underbrace{d\alpha_{U\cap V} \cap \psi}_{\text{wts } =0} - \alpha_{U\cap V} \cap d\psi\Big)$$

$$d(\alpha_{U\cap V} + \alpha_{U\setminus L} + \alpha_{V\setminus K}) = 0 \text{ in } C_*(M, M\setminus(K\cup L))$$

$\Rightarrow d\alpha_{U\cap V} \cap \psi = -\underbrace{d\alpha_{U\setminus L} \cap \psi}_{0} - \underbrace{d\alpha_{V\setminus K} \cap \psi}_{0}$ recall: $\psi|_{C_*(M\setminus K)} = 0$, $\psi|_{C_*(M\setminus L)} = 0$.

$\Rightarrow$ $\Rightarrow \alpha_{U\cap V} \cap d\psi = (-1)^{\dots} d(\alpha_{U\cap V} \cap \psi)$.

(Note that this step is missing in Hatcher.)

So from now on we work with $[\alpha_{U\cap V} \cap d\varphi_A]$.

$$d(\alpha_{U\cap V} \cap \varphi_A) = (-1)^{\ell} \cdot \left(d\alpha_{U\cap V} \cap \varphi_A - \alpha_{U\cap V} \cap d\varphi_A \right)$$

So we have

$$[\alpha_{U\cap V} \cap d\varphi_A] = \underbrace{[d\alpha_{U\cap V} \cap \varphi_A]} \quad \text{in } H_{n-\ell-1}(U\cap V), \qquad (*)$$

$$= \mu_{K\cap L} \cap \mathrm{exc} \circ \delta[\varphi]$$

Now we relate the homological and cohomological bdry to each other.

$$\partial(\mu_{K\cap L} \cap [\varphi]) = \partial\Big(\big[\underbrace{\alpha_{U\setminus L} \cap \varphi}_{\text{chain in } U} + \underbrace{\alpha_{U\cap V} \cap \varphi + \alpha_{V\setminus K} \cap \varphi}_{\text{chain in } V}\big]\Big) =$$

$$= [d(\alpha_{U\setminus L} \cap \varphi)] \underset{d\varphi=0}{=} (-1)^{\ell}[d\alpha_{U\setminus L} \cap \varphi] = \qquad \varphi = \varphi_A - \varphi_B$$

$$= (-1)^{\ell}[d\alpha_{U\setminus L} \cap \varphi_A] =$$

$\alpha_{U\cap V} + \alpha_{U\setminus L}$ represents μ_K $\Rightarrow$ $d(\alpha_{U\cap V} + \alpha_{U\setminus L})$ is a chain in $M\setminus K$

$$= (-1)^{\ell}[-d\alpha_{U\cap V} \cap \varphi_A] = (-1)^{\ell+1}[d\alpha_{U\cap V} \cap \varphi_A],$$

which is the same as in $(*)$ and the sign is as stated in the assertion. □

Now we finally prove Poincaré duality (Hatcher, Milnor – Stasheff):

$$D_M : H^{\ell}_c(M; R) \longrightarrow H_{n-\ell}(M; R)$$

is an iso for any R-orientable n-manifold M.

PROOF: Case 1. $M = \mathbb{R}^n$

$$D: H^{\ell}_c(\mathbb{R}^n) \longrightarrow H_{n-\ell}(\mathbb{R}^n)$$

If $\ell \neq n$, both sides are zero.

If $\ell = n$:

$$\begin{tikzcd} R \cong H^n_c(\mathbb{R}^n) \arrow[r, "D"] & H_0(\mathbb{R}^n) \cong R \\ H^n(\mathbb{R}^n, \mathbb{R}^n \setminus B) \arrow[u, "\cong"] \arrow[ur, "\mu_B \cap -"'] \arrow[u, phantom, shift left=6, "\circlearrowleft"] & \end{tikzcd}$$

This holds for any closed ball $B \subset \mathbb{R}^n$.

$\Rightarrow$ Sts: $\mu_B \cap -$ is an iso.

The Kronecker product $H^0(\mathbb{R}^n; R) \underset{R}{\otimes} H_0(\mathbb{R}^n; R) \xrightarrow{\langle -,- \rangle} R$ is non-degenerate.

$\Leftrightarrow$ $R \underset{R}{\otimes} R \longrightarrow R$ nd $\Leftrightarrow$ $H^0(\mathbb{R}^n; R) \xrightarrow{\cong} \mathrm{Hom}_R(H_0(\mathbb{R}^n; R), R)$

$$H^n(\mathbb{R}^n, \mathbb{R}^n \setminus B; R) \underset{\cong}{\xrightarrow{UCT}} \mathrm{Hom}_R(H_n(\mathbb{R}^n, \mathbb{R}^n \setminus B; R), R)$$

$\exists ! \; x \in H^n(\mathbb{R}^n, \mathbb{R}^n \setminus B; R)$ which is dual to μ_B $\Leftrightarrow$ $\langle x, \mu_B \rangle = 1$

Sts $\mu_B \cap x$ is a generator: $\langle 1, \mu_B \cap x \rangle = \langle x, \mu_B \rangle = 1 \Rightarrow \mu_B \cap x$ is a generator ✓

Case 2. $M = \bigcup_{\alpha \in L} U_\alpha$ where the U_α are open and PD holds for all U_α, and $\alpha \le \beta \Rightarrow U_\alpha \subseteq U_\beta$.

$$\begin{tikzcd}
H^\ell_c(M) \arrow[r, "D_M"] & H_{n-\ell}(M) \\
H^\ell_c(U_\alpha) \arrow[u] \arrow[r, "D_{U_\alpha}", "\cong"'] \arrow[ur, phantom, "\circlearrowleft"] & H_{n-\ell}(U_\alpha) \arrow[u]
\end{tikzcd} \quad \forall \alpha \in L$$

This diagram commutes: follows from the defs of the arrows.

Going to directed colims along $\alpha \in L$:

$$\begin{tikzcd}
H^\ell_c(M) \arrow[r, "D_M"] & H_{n-\ell}(M) \\
\operatorname{colim}_{\alpha \in L} H^\ell_c(U_\alpha) \arrow[u] \arrow[r, "\operatorname{colim}_{\alpha \in L} D_{U_\alpha}", "\cong"'] \arrow[ur, phantom, "\circlearrowleft"] & \operatorname{colim}_{\alpha \in L} H_{n-\ell}(U_\alpha) \arrow[u]
\end{tikzcd}$$

L totally ordered $\Rightarrow$ every cpt in M is contained in some U_α

$\Rightarrow$ the vertical maps are isos

$\Rightarrow$ D_M is an iso ✓

Case 3. $M = U \cup V$, PD holds for the open sets U, V and $U \cap V$

Follows from the 5-lemma and the diagram from before involving duality maps (p. 63)

Case 4. $M \subseteq \mathbb{R}^n$ any open subset.

If M is convex: $M \cong \mathbb{R}^n$, use Case 1. ✓

Any open can be written as a countable union $M = V_1 \cup V_2 \cup \dots$ where all the V_i are convex.

$M = \bigcup_{i=0}^{\infty}\left(\bigcup_{k=1}^{i} V_k\right)$ Case 3 $\Rightarrow$ PD holds for $\bigcup_{k=1}^{i} V_k$ by induction

Case 2 $\Rightarrow$ PD holds for M ✓

Case 5. M is arbitrary.

Consider the poset of open subsets of M. This is inductive, i.e. satisfies the condition of Zorn's lemma by Case 2.

Zorn's lemma $\Rightarrow$ $\exists$ maximal open U satisfying PD. Goal: $U = M$.

$\nleftrightarrow$ $U \subsetneq M$. $\Rightarrow \exists x \in M \setminus U$. Take $V \subseteq M$ to be an open nbhd of x s.t. $V \cong \mathbb{R}^n$.

Both U and V satisfy PD (Case 4) $\Rightarrow$ $U \cup V$ satisfies PD (Case 3), $U \subsetneq U \cup V$ ↯ $\square$

06.06.2018

Let M be a compact n-dimensional manifold.

$\forall H_i(M;\mathbb{Z})$ is finitely generated because $\exists N$ s.t. there is an embedding $M \hookrightarrow \mathbb{R}^N$ with M being a neighbourhood retract, and hence M has homology groups of a CW complex. (This has been mentioned before; we will treat it as a black box. The proof is mostly point set topology.)

Prop. M cpt. odd dimensional manifold $\Rightarrow \chi(M) = 0$.

From the previous observation it follows that $H_i(M;\mathbb{Z}) \cong \mathbb{Z}^{r_i} \oplus \bigoplus_j \mathbb{Z}/m_j$ (fund. thm. of fin. gen. ab. gps.)

where $r_i =: \operatorname{rk} H_i(M;\mathbb{Z}) = \dim_{\mathbb{Q}}(H_i(M;\mathbb{Z}) \otimes \mathbb{Q})$. This allows us to define

Def. $\chi(M) := \sum_{i=0}^{+\infty} (-1)^i \operatorname{rk} H_i(M;\mathbb{Z}) = \sum_{i=0}^{n} (-1)^i \operatorname{rk} H_i(M;\mathbb{Z})$. Euler characteristic

This agrees with our previous definition of χ for CW complexes.

Ex. $\chi(S^2) = \operatorname{rk} H_0 + \operatorname{rk} H_2 = 2$

PF OF PROP: Case 1. M is orientable.

$\Rightarrow \operatorname{rk} H_i(M;\mathbb{Z}) = \operatorname{rk} H^{n-i}(M;\mathbb{Z})$ by PD

UCT $\Rightarrow H^{n-i}(M;\mathbb{Z}) \cong \operatorname{Hom}_{\mathbb{Z}}(H_{n-i}(M;\mathbb{Z}), \mathbb{Z}) \oplus \operatorname{Ext}^1_{\mathbb{Z}}(H_{n-i-1}(M;\mathbb{Z}), \mathbb{Z})$

In general, for a finitely generated A the ext group $\mathrm{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z})$ is always torsion (and all higher ext groups vanish), because

$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}^r, A) = 0$ and $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m, B) \cong B/mB$.

$\Rightarrow \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-i-1}(M;\mathbb{Z}), \mathbb{Z})$ is torsion $\Rightarrow$ has $\mathrm{rk} = 0$.

$$\Rightarrow \chi(M) = \sum_{i=0} (-1)^i \,\mathrm{rk}\, H_i(M;\mathbb{Z}) = \underbrace{\mathrm{rk}\, H_0 - \underbrace{\mathrm{rk}\, H_1 + \ldots + \mathrm{rk}\, H_{n-1}}_{=} - \mathrm{rk}\, H_n}_{=} = 0$$

Case 2. M is not orientable.

M has PD with $\mathbb{F}_2$-coefficients $\Rightarrow$ we can give a similar argument, even easier because $\mathrm{Ext}^1_{\mathbb{F}_2} = 0$

$$\Rightarrow \underbrace{\chi(M, \mathbb{F}_2)}_{\substack{\text{mod } 2 \\ \text{Euler characteristic}}} := \sum_{i=0}^{n} (-1)^i \dim H_i(M, \mathbb{F}_2) = 0$$

Now we need to return to $\chi(M)$: we will show

$$\sum_{i=0}^{n} (-1)^i \dim H_i(M;\mathbb{F}_2) = \sum_{i=0}^{n} (-1)^i \,\mathrm{rk}\, H_i(M;\mathbb{Z})$$

UCT $\Rightarrow H^i(M;\mathbb{F}_2) \cong \mathrm{Hom}_{\mathbb{F}_2}(H_i(M;\mathbb{F}_2), \mathbb{F}_2) \cong H_i(M;\mathbb{F}_2)^\vee$ where $\vee$ denotes dual

$\Rightarrow H^i(M;\mathbb{F}_2) \cong H_i(M;\mathbb{F})$.

(Throughout the whole proof, Patchkoria keeps making remarks about the isos involved: namely that sometimes they are not canonical (like at the splitting) or that they may be complicated and the only reason we may neglect this is that we are only interested in the ranks.)

$$H_i(M;\mathbb{Z}) \cong \mathbb{Z}^{r_i} \oplus \bigoplus_{l=1}^{N_i} \mathbb{Z}/m_l \oplus \bigoplus_{j=1}^{K_i} \mathbb{Z}/s_j \quad \text{where } 2 \mid m_l,\ 2 \nmid s_j$$

$$\Rightarrow \dim H^i(M;\mathbb{F}_2) \underset{\text{UCT}}{=} \dim \mathrm{Hom}_{\mathbb{Z}}(H_i(M;\mathbb{Z}), \mathbb{F}_2) + \dim \mathrm{Ext}^1_{\mathbb{Z}}(H_{i-1}(M;\mathbb{Z}), \mathbb{F}_2) =$$

$$= r_i + N_i + N_{i-1}$$

$$\sum_{i=0}^{n} (-1)^i \dim H_i(M;\mathbb{F}_2) = \sum_{i=0}^{n} (-1)^i \dim H^i(M;\mathbb{F}_2)$$

$$\underbrace{\qquad\qquad}_{0=\chi(M,\mathbb{F}_2)=} \quad = \sum_{i=0}^{n} (-1)^i (r_i + N_i + N_{i-1})$$

$$= \sum_{i=0}^{n} (-1)^i r_i + \underbrace{\sum_{i=0}^{n} (-1)^i (N_i + N_{i-1})}_{N_0 - (N_1+N_0) + \dots + (N_{n-1}+N_{n-2}) - (N_n + N_{n-1})}$$

$$= \chi(M) + N_n$$

$H_n(M;\mathbb{Z})$ is torsion free M is not orientable $\Rightarrow N_n = 0. \Rightarrow \chi(M)=0.$

Now let M be an n-dimensional, R-orientable, not necessarily connected manifold, $[M] \in H_n(M;R)$ its fundamental class in the following sense:

If $M = \coprod_\alpha M_\alpha$, $\quad H_n(M;R) = \bigoplus_\alpha \underbrace{H_n(M_\alpha;R)}_{\cong R} \cong \bigoplus_\alpha R$,

$[M] = ([M_\alpha])_\alpha \in H_n(M;R)$ where $[M_\alpha]$ is the fund class of the connected component $[M_\alpha]$.

$$H^k(M;R) \underset{R}{\otimes} H^{n-k}(M;R) \xrightarrow{\ \cup_M\ } R \qquad \text{intersection product}$$

$$\varphi \otimes \psi \longrightarrow \langle \varphi \cup \psi, [M] \rangle$$

Recall that $\varphi \cup \psi \in H^{k+n-k}(M;R) = H^n(M;R)$ and

$$H^n(M;R) \underset{R}{\otimes} H_n(M;R) \xrightarrow{\langle -,- \rangle} R.$$

One may check that $\cup_M$ is well-defined on the tensor product.

We are only interested in the cases $R=\mathbb{Z}$ and $R=$ field.

For $R=\mathbb{Z}$: $\quad H^k(M;\mathbb{Z})_{free} := H^k(M;\mathbb{Z})/_{torsion}$

$$\begin{tikzcd} H^k(M;\mathbb{Z}) \underset{\mathbb{Z}}{\otimes} H^{n-k}(M;\mathbb{Z}) \arrow[r, "\cup_M"] \arrow[d] & \mathbb{Z} \\ H^k(M;\mathbb{Z})_{free} \otimes H^{n-k}(M;\mathbb{Z})_{free} \arrow[ur, dashed, "\exists!\, \cup_M"'] & \end{tikzcd}$$

We want to work with the free part because then we may use the UCT effectively; in the torsion, things just die.

Prop. Let M be an R-orientable compact n-mf. Then

1) $\cup_M^* : H^{n-k}(M;R) \xrightarrow{\cong} \mathrm{Hom}_R(H^k(M;R),R)$ if R is a field.

2) $\cup_M^* : H^{n-k}(M;\mathbb{Z})_{free} \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Z}}(H^k(M;\mathbb{Z})_{free},\mathbb{Z})$.

Rmk. In Topology I we have shown that $\chi(M)$ is independent from the choice of the field for CW complexes, but this result of course could not be applied for the general class of top. manifolds.

Pf: $H^{n-k}(M;R) \xrightarrow{h} \mathrm{Hom}_R(H_{n-k}(M;R),R) \xrightarrow{D^*} \mathrm{Hom}_R(H^k(M;R),R)$

$\psi \longmapsto \langle\psi,-\rangle \longmapsto \langle\psi,[M]\cap -\rangle$

since the duality map $H^k(M;R) \xrightarrow{D} H_{n-k}(M;R)$ is given by $[M]\cap -$.

$\varphi \longmapsto \langle\psi,[M]\cap\varphi\rangle = \langle\varphi\cup\psi,[M]\rangle$ by the cap-cup formula

$\cup_M^* := D^* \circ h$.

1) If R is a field: h is an iso by UCT, D is an iso by PD

$\Rightarrow \cup_M^*$ is an iso.

2)
$$\begin{tikzcd} H^{n-k}(M;\mathbb{Z})_{free} \arrow[r, "h", "\cong"'] \arrow[drr, bend right, "\cup_M^*"] & \mathrm{Hom}_{\mathbb{Z}}(H_{n-k}(M;\mathbb{Z}),\mathbb{Z}) \arrow[r, "D^*", "\cong"'] & \mathrm{Hom}_{\mathbb{Z}}(H^k(M;\mathbb{Z}),\mathbb{Z}) \arrow[d, phantom, "\cong"] \\ & & \mathrm{Hom}_{\mathbb{Z}}(H^k(M;\mathbb{Z})_{free},\mathbb{Z}) \end{tikzcd}$$
□

Cor. M cpt ori n-dim connected mf. Suppose $\alpha\in H^k(M;\mathbb{Z})$ is of infinite order and α is not a proper multiple of any element, i.e. if $\exists m\in\mathbb{Z}$, $\gamma\in H^k(M;\mathbb{Z})$ s.t. $\alpha = m\gamma$ then $m=\pm 1$.

Then $\exists\beta\in H^{n-k}(M;\mathbb{Z})$ such that $\alpha\cup\beta\in H^n(M;\mathbb{Z})\cong\mathbb{Z}$ is a generator

Note that β is only unique up to sign and torsion.

Pf: $H^k(M;\mathbb{Z})$ is finitely generated. $\Rightarrow H^k(M;\mathbb{Z}) \cong \underbrace{\mathbb{Z}\langle\alpha\rangle}_{\mathbb{Z}\text{ generated by }\alpha} \oplus B$

$\Rightarrow \exists\varphi : H^k(M;\mathbb{Z}) \longrightarrow \mathbb{Z}$ s.t. $\varphi(\alpha)=1$.

By the Prop.: $\exists \beta \in H^{n-k}(M;\mathbb{Z})$ s.t. $U_M^*(\beta) = \varphi$.

$$U_M^*(\beta) = \langle - \cup \beta, [M] \rangle = \varphi(-) \quad \Rightarrow \quad \langle \alpha \cup \beta, [M] \rangle = \varphi(\alpha) = 1.$$

$$H^n(M;\mathbb{Z}) \xrightarrow{\cong} \mathrm{Hom}\left(H_n(M;\mathbb{Z}), \mathbb{Z}\right)$$ Recall that the torsion of $H_{n-1}(M;\mathbb{Z})$ is 0.

$$\alpha \cup \beta \longmapsto \underbrace{\langle \alpha \cup \beta, - \rangle}_{\text{this is a generator}} \Rightarrow \alpha \cup \beta \text{ is a generator}$$

Let $F = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ (but not $\mathbb{O}$)

$$FP^n := F^{n+1} \setminus \{0\} \Big/ (x_0, \dots, x_n) \sim (y_0, \dots, y_n) \Leftrightarrow \exists \lambda \in F^\times \text{ s.t. } \lambda x_i = y_i \ \forall i = 0, \dots, n$$ projective space

The topology is the quotient topology from $F^{n+1} \setminus \{0\}$

FP^n are n-dimensional cpt connected manifolds

$U_i \subseteq FP^n \quad U_i = \{[x_0, \dots, x_n] \in FP^n \mid x_i \neq 0\}, \quad F^n \xrightarrow{\cong} U_i$

$$(x_1, \dots, x_n) \longmapsto [x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n]$$

$FP^n = U_0 \cup \dots \cup U_n$

Let $S(F^{n+1})$ be the unit sphere in F^{n+1}. Then the projection

$S(F^{n+1}) \twoheadrightarrow FP^n$ is surjective: $[x_0, \dots, x_n] = \left[\frac{x_0}{\|x\|}, \dots, \frac{x_n}{\|x\|}\right]$ where $\|x\| = \sqrt{\sum x_i^2}$

$F = \mathbb{R}$: $\mathbb{R}P^n$ is a CW complex

$$\begin{array}{ccc} S^n & \longrightarrow & \mathbb{R}P^n \\ \downarrow & & \downarrow \\ D^n & \longrightarrow & \mathbb{R}P^{n+1} \end{array}$$

$$H_n(\mathbb{R}P^n;\mathbb{Z}) = \begin{cases} 0 & n \geq 2, \ 2 \mid n \\ \mathbb{Z} & 2 \nmid n \end{cases}$$ $\Rightarrow \mathbb{R}P^n$ is orientable if $2 \nmid n$ and non-orientable if $2 \mid n$.

$$H_i(\mathbb{R}P^n, \mathbb{F}_2) = \begin{cases} 0 & i > n \\ \mathbb{F}_2 & 0 \leq i \leq n \end{cases} \Rightarrow H^i(\mathbb{R}P^n, \mathbb{F}_2) = \begin{cases} 0 & i > n \\ \mathbb{F}_2 & 0 \leq i \leq n \end{cases}$$

$\underset{\sim}{F = \mathbb{C}}$:

$$\begin{tikzcd} S^{2n+1} \arrow[r, two heads] \arrow[d] & \mathbb{C}P^n \arrow[d] \\ D^{2n+2} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathbb{C}P^{n+1} \end{tikzcd}$$

CW complex with no odd dimensional cells

$$H_i(\mathbb{C}P^n; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & 0 \leq i \leq 2n,\ 2 \mid i \\ 0 & \text{else} \end{cases}$$

$$\text{UCT} \Rightarrow H^i(\mathbb{C}P^n; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & 0 \leq i \leq 2n,\ 2 \mid i \\ 0 & \text{else} \end{cases}$$

$\underset{\sim}{F = \mathbb{H}}$:

$$\begin{tikzcd} S^{4n+3} \arrow[r] \arrow[d] & \mathbb{H}P^n \arrow[d] \\ D^{4n+4} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathbb{H}P^{n+1} \end{tikzcd}$$

$$H_i(\mathbb{H}P^n; \mathbb{Z}) \cong H^i(\mathbb{H}P^n; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & 0 \leq i \leq 4n,\ 4 \mid i \\ 0 & \text{else} \end{cases}$$

What about $\mathbb{O}$? The projective spaces $\mathbb{O}P^1 = S^8$ and $\mathbb{O}P^2$ (Cayley plane) can be defined. The relation won't be an eq. relation. Of course it can be made into one but it won't yield a manifold.

Next goal: compute the cohomology rings, using the Cor. (p. 72).